

LOG-CONCAVITY OF COMPLEXITY ONE HAMILTONIAN TORUS ACTIONS

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ABSTRACT. Let (M, ω) be a closed $2n$ -dimensional symplectic manifold equipped with a Hamiltonian T^{n-1} -action. Then Atiyah-Guillemin-Sternberg convexity theorem implies that the image of the moment map is an $(n-1)$ -dimensional convex polytope. In this paper, we show that the density function of the Duistermaat-Heckman measure is log-concave on the image of the moment map.

1. INTRODUCTION

In statistic physics, the relation $S(E) = k \log W(E)$ is called Boltzmann's principle where W is the number of states with given values of macroscopic parameters E (like energy, temperature, ..), k is the Boltzmann's constant, and S is the entropy of the system which measures the degree of disorder in the system. For the additive values E , it is well-known the entropy is always concave function. (See [O1] for more detail). In symplectic setting, consider a Hamiltonian G -manifold (M, ω) with the moment map $\mu : M \rightarrow \mathfrak{g}^*$. The Liouville measure m_L is defined by

$$m_L(U) := \int_U \frac{\omega^n}{n!}$$

for any open set $U \subset M$. Then the push-forward measure $m_{\text{DH}} := \mu_* m_L$, called the *Duistermaat-Heckman measure*, can be regarded as a measure on \mathfrak{g}^* such that for any Borel subset $B \subset \mathfrak{g}^*$, $m_{\text{DH}}(B) = \int_{\mu^{-1}(B)} \frac{\omega^n}{n!}$ tells us that how many states of our system have momenta in B . By the Duistermaat-Heckman theorem [DH], m_{DH} can be expressed in terms of the density function $\text{DH}(\xi)$ with respect to the Lebesgue measure on \mathfrak{g}^* . Therefore the concavity of the entropy of given Hamiltonian system on (M, ω) can be interpreted as the log-concavity of $\text{DH}(\xi)$ on the image of μ . A. Okounkov [O2] proved that the density function of the Duistermaat-Heckman measure is log-concave on the image of the moment map for the maximal torus action when (M, ω) is the co-adjoint orbit of some classical Lie groups. In [Gr], W. Graham showed the log-concavity of the density function of the Duistermaat-Heckman measure also holds for any Kähler manifold admitting a holomorphic Hamiltonian torus action. V. Ginzberg and A. Knudsen conjectured independently that the log-concavity holds for any Hamiltonian G -manifolds, but it turns out to be false in general by Y. Karshon [K1]. Further related works can be found in [L] and [C].

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As noted in [K1] and [Gr], log-concavity holds for Hamiltonian toric (i.e. complexity zero) actions, and Y. Lin dealt with log-concavity of complexity two Hamiltonian torus actions in [L]. But, there is no result on log-concavity of complexity one Hamiltonian torus actions. This is why we restrict our interest to complexity one. From now on, we assume that (M, ω) is a $2n$ -dimensional closed symplectic manifold with an effective Hamiltonian T^{n-1} -action. Let $\mu : M \rightarrow \mathfrak{t}^*$ be the corresponding moment map where \mathfrak{t}^* is a dual of the Lie algebra of T^{n-1} . By Atiyah-Guillemin-Sternberg convexity theorem, the image of the moment map $\mu(M)$ is an $(n-1)$ -dimensional convex polytope in \mathfrak{t}^* . By the Duistermaat-Heckman theorem [DH], we have

$$m_{\text{DH}} = \text{DH}(\xi) d\xi$$

where $d\xi$ is the Lebesgue measure on $\mathfrak{t}^* \cong \mathbb{R}^{n-1}$ and $\text{DH}(\xi)$ is a continuous piecewise polynomial function of degree less than 2 on \mathfrak{t}^* . Our main theorem is as follow.

Theorem 1.1. *Let (M, ω) be a $2n$ -dimensional closed symplectic manifold equipped with a Hamiltonian T^{n-1} -action with the moment map $\mu : M \rightarrow \mathfrak{t}^*$. Then the density function of the Duistermaat-Heckman measure is log-concave on $\mu(M)$.*

2. PROOF OF THE THEOREM 1.1

Let (M, ω) be a $2n$ -dimensional closed symplectic manifold. Let $(n-1)$ -dimensional torus T acts on (M, ω) in Hamiltonian fashion. Denote by \mathfrak{t} the Lie algebra of T . For a moment map $\mu : M \rightarrow \mathfrak{t}^*$ of the T -action, define the Duistermaat-Heckman function $\text{DH} : \mathfrak{t}^* \rightarrow \mathbb{R}$ as

$$\text{DH}(\xi) = \int_{M_\xi} \omega_\xi$$

where M_ξ is the reduced space $\mu^{-1}(\xi)/T$ and ω_ξ is the corresponding reduced symplectic form on M_ξ .

Now, we define the x-ray of our action. Let T_1, \dots, T_N be the subgroups of T^{n-1} which occur as stabilizers of points in M^{2n} . Let M_i be the set of points whose stabilizers are T_i . By relabeling, we can assume that M_i 's are connected and the stabilizer of points in M_i is T_i . Then, M^{2n} is a disjoint union of M_i 's. Also, it is well known that M_i is open dense in its closure and the closure is just a component of the fixed set M^{T_i} . Let \mathfrak{M} be the set of M_i 's. Then, the *x-ray* of (M^{2n}, ω, μ) is defined as the set of $\mu(\overline{M_i})$'s. Here, we recall a basic lemma.

Lemma 2.1. [GS, Theorem 3.6] *Let \mathfrak{h} be the Lie algebra of T_i . Then $\mu(M_i)$ is locally of the form $x + \mathfrak{h}^\perp$ for some $x \in \mathfrak{t}^*$.*

By this lemma, $\dim_{\mathbb{R}} \mu(M_i) = m$ for $(n-1-m)$ -dimensional T_i . Each image $\mu(\overline{M_i})$ (resp. $\mu(M_i)$) is called an *m-face* (resp. an *open m-face*) of the x-ray if T_i is $(n-1-m)$ -dimensional. Our interest is mainly in open $(n-2)$ -faces of the x-ray, i.e. codimension one in \mathfrak{t}^* . Figure 2.1 is an example of x-ray with $n=3$ where thick lines are $(n-2)$ -faces. Now, we can prove the main theorem.

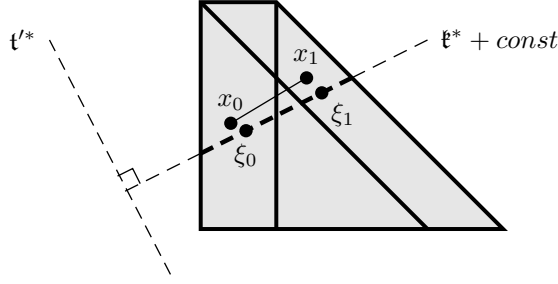


FIGURE 2.1. Proof of Theorem 1.1

Proof of Theorem 1.1. When $n = 2$, we obtain a proof by [K2, Lemma 2.19]. So, we assume $n \geq 3$. Pick arbitrary two points x_0, x_1 in the image of μ . We should show that

$$(2.1) \quad t \log (\mathrm{DH}(x_1)) + (1 - t) \log (\mathrm{DH}(x_0)) \leq \log (\mathrm{DH}(tx_1 + (1 - t)x_0))$$

for each $t \in [0, 1]$. Put $x_t = tx_1 + (1 - t)x_0$.

Let us fix a decomposition $T = S^1 \times \cdots \times S^1$. By the decomposition, we identify \mathfrak{t} with \mathbb{R}^{n-1} , and \mathfrak{t} carries the usual Riemannian metric $\langle \cdot, \cdot \rangle_0$ which is a bi-invariant metric. This metric gives the isomorphism

$$\iota : \mathfrak{t} \rightarrow \mathfrak{t}^*, \quad X \mapsto \langle \cdot, X \rangle_0.$$

For a small $\epsilon > 0$, pick two regular values ξ_i in the ball $B(x_i, \epsilon)$ for $i = 0, 1$ which satisfy the following two conditions:

- i. $\xi_1 - \xi_0 \in \iota(\mathbb{Q}^{n-1})$,
- ii. the line L containing ξ_0, ξ_1 in \mathfrak{t}^* meets each open m -face transversely for $m = 1, \dots, n - 2$.

Transversality guarantees that the line does not meet any open m -face for $m \leq n - 3$. Put

$$\xi_t = t\xi_1 + (1 - t)\xi_0 \text{ and } X = \iota^{-1}(\xi_1 - \xi_0).$$

Let $\mathfrak{k} \subset \mathfrak{t}$ be the one-dimensional subalgebra spanned by X . By i., \mathfrak{k} becomes a Lie algebra of a circle subgroup of T , call it K . Let \mathfrak{t}' be the orthogonal complement of \mathfrak{k} in \mathfrak{t} . Again by i., \mathfrak{t}' becomes a Lie subalgebra of a $(n - 2)$ -dimensional subtorus of T , call it T' . Let

$$p : \mathfrak{t}^* \rightarrow \mathfrak{t}'^* = \iota(\mathfrak{t}')$$

be the orthogonal projection along $\mathfrak{k}^* = \iota(\mathfrak{k})$. If we put $\mu' = p \circ \mu$, then $\mu' : M \rightarrow \mathfrak{t}'^*$ is a moment map of the restricted T' -action on M . Put $\xi'_t = p(\xi_t)$ for $t \in [0, 1]$.

We would show that ξ'_t is a regular value of μ' . For this, we show that each point $x \in \mu'^{-1}(\xi'_t)$ is a regular point of μ' . By ii. and Lemma 2.1, stabilizer T_x is finite or one-dimensional. If T_x is finite, then x is a regular point of μ so that it is also a regular point of μ' . If T_x is one-dimensional, then $\mu(x)$ is a point of an open $(n - 2)$ -face $\mu(M_i)$ such that $x \in M_i$. Let \mathfrak{h} be the Lie algebra of $T_i = T_x$. By Lemma 2.1, $p(d\mu(T_x M_i)) = p(\mathfrak{h}^\perp)$, and the kernel \mathfrak{k} of p is not contained in \mathfrak{h}^\perp by transversality. So, $p(\mathfrak{h}^\perp)$ is the whole \mathfrak{t}'^* because

$\dim \mathfrak{h}^\perp = \dim \mathfrak{t}^*$, and this means that x is a regular point of μ' . Therefore, we have shown that ξ' is a regular value of μ' .

Since ξ' is a regular value, the preimage $\mu'^{-1}(\xi')$ is a manifold and T' acts almost freely on it, i.e. stabilizers are finite. So, if we denote by $M_{\xi'}$ the symplectic reduction $\mu'^{-1}(\xi')/T'$, then it becomes a symplectic orbifold carrying the induced symplectic T/T' -action. We can observe that the image of $\mu'^{-1}(\xi')$ through μ is the thick dashed line in Figure 2.1. Since $K/(K \cap T') \cong T/T'$, we will regard $K/(K \cap T')$ and \mathfrak{k} as T/T' and its Lie algebra, respectively. The map $\mu_X := \langle \mu, X \rangle$ induces a map on $M_{\xi'}$ by T -invariance of μ , call it just μ_X where $\langle \cdot, \cdot \rangle : \mathfrak{t}^* \times \mathfrak{t} \rightarrow \mathbb{R}$ is the evaluation pairing. Then, we can observe that μ_X is a Hamiltonian of the $K/(K \cap T')$ -action on $M_{\xi'}$, and that M_{ξ_t} is symplectomorphic to the symplectic reduction of $M_{\xi'}$ at the regular value $\langle \xi_t, X \rangle$ with respect to μ_X . If we denote by DH_X the Duistermaat-Heckman function of $\mu_X : M_{\xi'} \rightarrow \mathbb{R}$, then we have $\text{DH}(\xi_t) = \text{DH}_X(\langle \xi_t, X \rangle)$ for $t \in [0, 1]$. Since $M_{\xi'}$ is a four-dimensional symplectic orbifold with Hamiltonian circle action, DH_X is log-concave by Lemma 2.2 below. Since x_t and ξ_t are sufficiently close and DH is continuous by [DH], we can show (2.1) by log-concavity of DH_X . \square

Lemma 2.2. *Let (N, σ) be a closed four dimensional Hamiltonian S^1 -orbifold. Then the density function of the Duistermaat-Heckman measure is log-concave.*

Proof. Let $\phi : N \rightarrow \mathbb{R}$ be a moment map. Then the density function $\text{DH} : \text{Im}\phi \rightarrow \mathbb{R}_{\geq 0}$ of the Duistermaat-Heckman measure is given by

$$\text{DH}(t) = \int_{N_t} \sigma_t$$

for any regular value $t \in \text{Im}\phi$. Let $(a, b) \subset \text{Im}\phi$ be an open interval consisting of regular values of ϕ and fix $t_0 \in (a, b)$. By the Duistermaat-Heckman theorem [DH], $[\sigma_t] - [\sigma_{t_0}] = -e(t - t_0)$ for any $t \in (a, b)$, where e is the Euler class of the S^1 -fibration $\phi^{-1}(t_0) \rightarrow \phi^{-1}(t_0)/S^1$. Therefore

$$\text{DH}'(t) = - \int_{N_t} e$$

and

$$\text{DH}''(t) = 0$$

for any $t \in (a, b)$. Note that $\text{DH}(t)$ is log-concave on (a, b) if and only if it satisfies $\text{DH}(t) \cdot \text{DH}''(t) - \text{DH}'(t)^2 \leq 0$ for all $t \in (a, b)$. Hence $\text{DH}(t)$ is log-concave on any open intervals consisting of regular values.

Let c be any interior critical value of ϕ in $\text{Im}\phi$. Then it is enough to show that the jump in the derivative of $(\log \text{DH})'$ is negative at c . First, we will show that the jump of the value $\text{DH}'(t) = - \int_{N_t} e$ is negative at c . Choose a small $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon)$ does not contain a critical value except for c . Let N_c be a symplectic cut of $\phi^{-1}[c - \epsilon, c + \epsilon]$ along the extremum so that N_c becomes a closed Hamiltonian S^1 -orbifold whose maximum is the reduced space $M_{c+\epsilon}$ and the minimum is $N_{c-\epsilon}$. Using the Atiyah-Bott-Berline-Vergne localization formula for orbifolds [M], we have

$$0 = \int_{N_c} 1 = \sum_{p \in N^{S^1} \cap \phi^{-1}(c)} \frac{1}{d_p} \frac{1}{p_1 p_2 \lambda^2} + \int_{M_{c-\epsilon}} \frac{1}{\lambda + e_-} + \int_{N_{c+\epsilon}} \frac{1}{-\lambda - e_+}$$

which is equivalent to

$$0 = \sum_{p \in N^{S^1} \cap \phi^{-1}(c)} \frac{1}{p_1 p_2} = \int_{N_{c-\epsilon}} e_- - \int_{N_{c+\epsilon}} e_+,$$

where d_p is the order of the local group of p , p_1 and p_2 are the weights of the tangential S^1 -representation on $T_p N$, and e_- (e_+ respectively) is the Euler class of $\phi^{-1}(c - \epsilon)$ ($\phi^{-1}(c + \epsilon)$ respectively). Since c is in the interior of $\text{Im} \phi$, we have $p_1 p_2 < 0$ for any $p \in N^{S^1} \cap \phi^{-1}(c)$. Hence the jump of $\text{DH}'(t) = -\int_{N_t} e$ is negative at c , which implies that the jump of $\log \text{DH}(t)' = \frac{\text{DH}'(t)}{\text{DH}(t)}$ is negative at c (by continuity of $\text{DH}(t)$). It finishes the proof. \square

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